

**12<sup>th</sup> Annual Johns Hopkins Math Tournament**  
**Saturday, February 19, 2011**

**Algebra Subject Test**

1. [1025] For each positive integer  $n$ , let  $i_n$  denote the exponential tower

$$\underbrace{\left( \left( \left( i^i \right)^i \right)^i \right)^i}_{n \text{ times}}$$

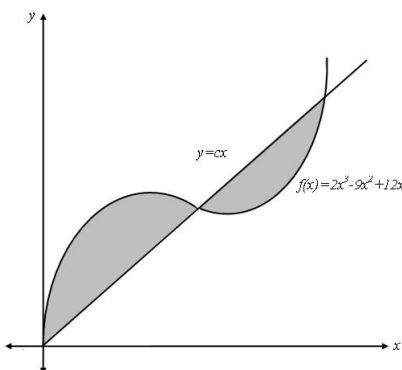
where  $i = \sqrt{-1}$ ; for example,  $i_1 = i$ ,  $i_2 = i^i$ , and  $i_3 = (i^i)^i$ . Find  $i_{2011}$ .

2. [1026] Consider the curves  $x^2 + y^2 = 1$  and  $2x^2 + 2xy + y^2 - 2x - 2y = 0$ . These curves intersect at two points, one of which is  $(1, 0)$ . Find the other one.
3. [1028] Let  $F(x)$  be a real-valued function defined for all real  $x \neq 0, 1$  such that

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$

Find  $F(2)$ .

4. [1032] Let  $p$  be a monic cubic polynomial such that the sum of the coefficients, the sum of the roots, and the sum of each root squared are all equal to 1. Find  $p$ . Note: monic means that the leading coefficient of  $p$  is 1.
5. [1040] The line  $y = cx$  is drawn such that it intersects the curve  $f(x) = 2x^3 - 9x^2 + 12x$  at two points in the first quadrant, creating the two shaded regions as shown in the diagram (not to scale). If the areas of the two shaded regions are the same, what is  $c$ ?



6. [1056] Let  $p(x) = (x^3 + x + 1)^{2011}$ . Let  $\omega = e^{2\pi i/5}$ . Compute  $p(\omega)p(\omega^2)p(\omega^3)p(\omega^4)$ .
7. [1080] Find all integers  $x$  for which  $|x^3 + 6x^2 + 2x - 6|$  is prime.
8. [1152] Find the final non-zero digit in  $100!$ .
9. [1280] Let  $T_n$  denote the number of terms in  $(x + y + z)^n$  when simplified, i.e. expanded and like terms collected, for non-negative integers  $n \geq 0$ . Find

$$\sum_{k=0}^{2010} (-1)^k T_k = T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}.$$

10. [1536] Define a sequence  $(a_n)$  by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= a_2 = \cdots = a_7 = 0 \\ a_n &= \frac{a_{n-8} + a_{n-7}}{2} \text{ for } n \geq 8 \end{aligned}$$

Find the limit of this sequence.